Lifting the Exponent

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Definition (p-adic valuation)**.** Let p be a prime number and n be an integer. We define $\nu_p(n)$, the p-adic valuation of n, to be the largest integer k such that $p^k|n$.

We may also write $p^k||n$ if $p^k|n$ and $p^{k+1} \nmid n$.

Theorem (Lifting the Exponent Lemma)**.** Let p be an odd prime. Suppose that x, y are integers not divisible by p such that $p|x-y$, then:

$$\nu_p(x^n - y^n) = \nu_p(x-y) + \nu_p(n)$$

Theorem (LTE for $p = 2$)**.** Let $p = 2$, let x, y be odd integers such that $4|x-y$, then:

$$\nu_2(x^n - y^n) = \nu_2(x-y) + \nu_2(n)$$

Corollary. If n is an even integer and x, y are any odd integers, then:

$$\nu_2(x^n - y^n) = \nu_2(x-y) + \nu_2(x+y) + \nu_2(n) - 1$$

Selected Problems

1. Prove the Lifting the Exponent Lemma.
2. Find all positive integers a such that $\dfrac{5^a+1}{3^a}$ is a positive integer.
3. Let $k > 1$ be an integer. Show that there exists infinitely many positive integers n such that
$$n|1^n + 2^n + 3^n + \cdots + k^n$$
4. (CGMO 2017)
   (a) Find all positive integers n such that for any odd integer a, we have $4 \mid a^n - 1$
   (b) Find all positive integers n such that for any odd integer a, we have $2^{2017} \mid a^n - 1$
5. (India 2018) For a natural number $k > 1$, define S_k to be the set of all triplets (n, a, b) of natural numbers, with n odd and $\gcd(a, b) = 1$, such that $a + b = k$ and n divides $a^n + b^n$. Find all values of k for which S_k is finite.
6. (Indonesia 2010) Let n be a positive integer with $n = p^{2010}q^{2010}$ for two odd primes p and q. Show that there exist exactly $\sqrt[2010]{n}$ positive integers $x \le n$ such that $p^{2010}|x^p - 1$ and $q^{2010}|x^q - 1$.

7. (CWMI 2010) Suppose that m and k are non-negative integers, and $p = 2^{2^m} + 1$ is a prime number. Prove that
   (a) $2^{2^{m+1}p^k} \equiv 1 \pmod{p^{k+1}}$;
   (b) $2^{m+1}p^k$ is the smallest positive integer n satisfying the congruence equation $2^n \equiv 1 \pmod{p^{k+1}}$.

8. (Romania 2024) Let n be a positive integer and let a and b be positive integers congruent to 1 modulo 4. Prove that there exists a positive integer k such that at least one of the numbers $a^k - b$ and $b^k - a$ is divisible by 2^n.

9. (Russia 1996) Find all natural numbers n, such that there exist relatively prime integers x and y and an integer $k > 1$ satisfying the equation $3^n = x^k + y^k$.

10. (Turkey 2024) Find all positive integer pairs (a, b) such that,

$$\frac{10^{a!} - 3^b + 1}{2^a}$$

is a perfect square.

11. (Balkan 2018) Find all primes p and q such that $3p^{q-1} + 1$ divides $11^p + 17^p$

12. (IMO 1990) Determine all integers $n > 1$ such that

$$\frac{2^n + 1}{n^2}$$

is an integer.

13. (CMO 2018) Let n be a positive integer. Let A_n denote the set of primes p such that there exists positive integers a, b satisfying

$$\frac{a+b}{p} \text{ and } \frac{a^n + b^n}{p^2}$$

are both integers that are relatively prime to p. If A_n is finite, let $f(n)$ denote $|A_n|$.
   (a) Prove that A_n is finite if and only if $n \neq 2$.
   (b) Let m, k be odd positive integers and let d be their gcd. Show that

$$f(d) \leq f(k) + f(m) - f(km) \leq 2f(d).$$