YY RANTs at YRANT VII:

A Trace-Path Integral formula over function fields

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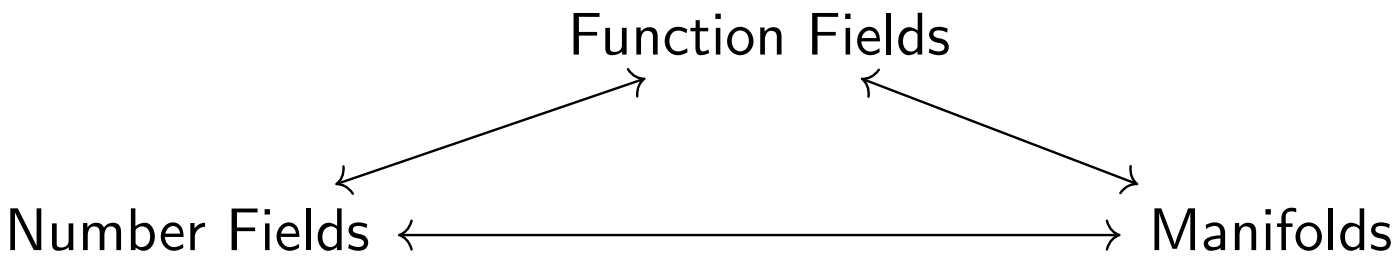
5 September 2025

Analogy, Analogy, Analogy



Function Fields and Manifolds

Let $X/\mathbb{F}_q$ be a smooth projective curve. $\bar{X} := X \times_{\mathbb{F}_q} \operatorname{Spec} \bar{\mathbb{F}}_q$ is analogous to a Riemann Surface Σ.

$$\begin{array}{ccc} \bar{X} & \hookrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \bar{\mathbb{F}}_q & \longrightarrow & \operatorname{Spec} \mathbb{F}_q \end{array} \qquad \begin{array}{ccc} \Sigma & \hookrightarrow & M \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & S^1 \end{array}$$

The manifold M can be viewed as a mapping torus

$$M = \Sigma \times [0,1]/F$$

where $F : \Sigma \to \Sigma$ is the monodromy action of the loop in S^1 on Σ

In a similar manner, we can view X as a fibred 3 manifold of the form

$$X \text{ ‘}=\text{ ’ } \bar{X} \times [0,1]/\operatorname{Fr}_q .$$

The minimum amount of physics to get my point across

M - Symplectic Manifold, usually $M = T^*X$ for some manifold X.

L - Lagrangian submanifold of M, usually just the zero sections of $M = T^*X$, so $L = X$.

$\mathcal{H}$ - Hilbert space of functions, obtained from a process called *quantisation*.

Ignoring constants, the Schrödinger's equation can be written as

$$\frac{d\psi}{dt} = -iH\psi,$$

where ψ is a time-dependent vector in $\mathcal{H}$, and H is a self-adjoint operator on $\mathcal{H}$ called the *Hamiltonian*.

The minimum amount of physics to get my point across

Solutions to the time dependent Schrödinger's equation can be written in the form

$$[e^{-iHT}\psi](x) = \int K_T(x,y)\psi(y)dy$$

for some kernel function $K_T(x,y)$.

The path integral formalism interprets $K_T(x,y)$ as an integral over paths

$$K_T(x,y) = \int_{P(x,y)} e^{iA(q)}dq,$$

where $A(q)$ is the classical action defined on paths $q : [0,T] \to L$, and $P(x,y)$ is the set of paths starting at y and ending at x.

The minimum amount of physics to get my point across

We can interpret the kernel function $K_T(x, y)$ as a 'matrix' with infinite dimensions, and under this viewpoint one has informally

$$\mathrm{Tr}(e^{-iHT}) = \int K_T(x, x)dx.$$

But substituting $K_T(x, y) = \int_{P(x,y)} e^{iA(q)}dq$, we get

$$\mathrm{Tr}(e^{-iHT}) = \iint_{P(x,x)} e^{iA(q)}dqdx = \int_{\Omega} e^{iA(q)}dq.$$

Where Ω is the space of all loops $S^1 \to L$.

The minimum amount of physics to get my point across

In fact, we can view the space of loops Ω as the space of sections of the trivial line bundle $L \times S^1 \to S^1$.

Suppose instead of the trival line bundle we instead had a bundle

$$Y = L \times [0, T]/F \to S^1$$

where the monodromy map $F : L \to L$ is used to glue $T \times L$ to $0 \times L$.

If we additionally assume that our theory is *topological* so that the Hamiltonian is equal to zero, then via a similar calculation (omitted due to time) there is a trace formula:

$$\mathrm{Tr}(F|\mathcal{H}) = \int_{\mathcal{F}} e^{-A(\gamma)} d\gamma.$$

Where $\mathcal{F}$ is the space of all sections of this twisted bundle $Y \to S^1$

Back to Arithmetic!

The main result today is an arithmetic analogue of a trace path integral formula in quantum field theory

$$\mathrm{Tr}(F|\mathcal{H}) = \int_{\mathcal{F}} e^{-A(\gamma)} d\gamma.$$

The arithmetic analogue is

Theorem (C., Kim-Venkatesh, 2025)

$$\mathrm{Tr}(\mathrm{Fr}_q\,|\mathcal{H}) = \pm \sum_{\gamma \in J[\ell](\mathbb{F}_q)} e^{iA(\gamma)}$$

We now explain how these objects are defined in arithmetic, and give some insight into how the trace and path integral are computed.

Arithmetic Path Integrals

Again, let $J/\mathbb{F}_q$ be the Jacobian of a smooth projective curve $X/\mathbb{F}_q$. By a similar diagram

$$\begin{array}{ccc} \bar{J}[\ell] & \hookrightarrow & J[\ell] \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\bar{\mathbb{F}}_q) & \longrightarrow & \mathrm{Spec}(\mathbb{F}_q) \end{array}$$

We can view $J[\ell]$ as a manifold fibred over the circle, with finite fibres.

Moreover, sections of the bundle $J[\ell] \to \mathrm{Spec}\,\mathbb{F}_q$ are maps γ

$$\begin{array}{c} J[\ell] \\ \downarrow \uparrow \gamma \\ \mathrm{Spec}\,\mathbb{F}_q. \end{array}$$

These are precisely the $\mathbb{F}_q$ rational points $J[\ell](\mathbb{F}_q)$.

Arithmetic Path Integral

Integral over space of sections $\mathcal{F} \rightsquigarrow$ Sum over $\mathbb{F}_q$ rational points $J[\ell](\mathbb{F}_q)$.

But what are we summing? Whats the action A?

We define A to be a pairing coming from class field theory.

(Assume $\mu_\ell \subset \mathbb{F}_q$ from now on, so μ_ℓ and $\frac{1}{\ell}\mathbb{Z}/\mathbb{Z}$ are isomorphic as Galois modules and sheaves.)

The Arithmetic Action A

An element $\gamma \in J[\ell](\mathbb{F}_q)$ defines a line bundle L_γ over X such that $(L_\gamma)^{\otimes \ell} \simeq \mathcal{O}_X$. Fix an isomorphism $f : (L_\gamma)^{\otimes \ell} \xrightarrow{\sim} \mathcal{O}_X$, then define a $\mu_\ell = \frac{1}{\ell}\mathbb{Z}/\mathbb{Z}$ torsor via

$$c_{\gamma,f}(U) := \left\{ y \in \Gamma(L_\gamma, U) : f(y^{\otimes \ell}) = 1 \right\} \in H^1(X, \tfrac{1}{\ell}\mathbb{Z}/\mathbb{Z})$$

On the other hand, there is a reciprocity map

$$\mathrm{Rec} : J(\mathbb{F}_q) \to \pi_1^{ab}(X)$$

Noting that $H^1(X, \frac{1}{\ell}\mathbb{Z}/\mathbb{Z}) = \mathrm{Hom}(\pi_1(X), \frac{1}{\ell}\mathbb{Z}/\mathbb{Z})$, we define

$$A(\gamma, \beta) := c_{\gamma,f}(\mathrm{Rec}(\beta)).$$

It can be shown that this pairing is independent of the choice for f.

Evaluating the integral

Setting $A(\gamma) := A(\gamma, \gamma)$, the path integral we wish to evaluate evaluates to

$$\sum_{\gamma \in J[\ell](\mathbb{F}_q)} e^{iA(\gamma)} = \sqrt{|J[\ell](\mathbb{F}_q)|} \left(\frac{(-1)^{(\dim_\ell J[\ell](\mathbb{F}_q))/2} \det(A)}{\ell} \right)$$

Key Ideas:

- The main ingredient is to show that $A(\beta, \gamma)$ is a symmetric bilinear form.
- Once A is shown to be a symmetric bilinear form, then $A(q)$ is a quadratic form. Then there are known theorems that can evaluate sums of the above form.
- In order to show A is symmetric, I showed that this pairing A is actually equal to a function field analogue of the Arithmetic Chern-Simons action (I talked a bit about this at YRANT last year).

Trace Side

What is the arithmetic analogue for the *quantisation* $\mathcal{H}$?

Given a manifold M with symplectic form ω, suppose we have a line bundle $\mathcal{L}$ with Chern class

$$c_1(\mathcal{L}) = \omega,$$

then the *geometric quantisation* of (M, ω) are the holomorphic sections

$$\mathcal{H}_k = \Gamma_{hol}(M, \mathcal{L}^{\otimes k}).$$

Luckily, torsions of Jacobians $J[\ell]$ already comes naturally equipped with a symplectic form — the Weil pairing.

Arithmetic Hilbert Space

Let Y be a lift of the curve $\bar{X}$ to the Witt vectors $W(\bar{\mathbb{F}}_q)$, and J_Y be the Jacobian of Y.

J_Y comes equipped with a canonical principal polarisation coming from the intersection pairing on Y. Let $\Theta \to J_Y$ be the theta line bundle associated to this polarisation.

Turns out the Chern class of Θ coincides with the Weil pairing up to sign (Orr-Skorobogatov-Zarhin, 2021), so we define the Hilbert space to be

$$\mathcal{H} := \Gamma(J_Y, \Theta^{\otimes \ell}) \otimes \mathbb{C}.$$

Computing the trace

The space $\mathcal{H}$ admits an action by the finite Heisenberg group

$$H(J[\ell]) := \mathbb{F}_\ell \times J[\ell]$$

with group action

$$(\lambda, a) \circ (\mu, b) = \left(\lambda + \mu + \tfrac{1}{2}\omega(a, b), a + b\right).$$

The space $\mathcal{H}$ is the unique (up to unique isomorphism) representation of $H(J[\ell])$ with identity central character. Using machinery from Gurevich-Hadani (2011), we can write out this representation explicitly via functions on Lagrangian subspaces on $J[\ell]$.

Then the action of Fr_q on $\mathcal{H}$ can also be written out explicitly, and we use that to compute the trace

$$\mathrm{tr}(\mathrm{Fr}_q \,|\mathcal{H}) = \left(\frac{(-1)^{n-1/2}(-1)^{(\deg \overline{\chi_{\mathrm{Fr}_q}})/2}\overline{\chi_{\mathrm{Fr}_q}}(1)}{\mathbb{F}_\ell}\right) \sqrt{|J[\ell](\mathbb{F}_q)|}.$$

So combining everything:

Theorem (C., Kim-Venkatesh, 2025)

Let J be the Jacobian of a genus g curve X over a finite field $\mathbb{F}_q$. For primes ℓ satisfying $q \equiv 1 \pmod{\ell}$, supposing that Fr_q acts semisimply on the $\mathbb{F}_\ell$ vector space $J[\ell]$, we have the equality

$$\mathrm{tr}(\mathrm{Fr}_q \,|\mathcal{H}) = \left(\frac{(-1)^g \overline{\chi_{\mathrm{Fr}_q}}(1)\det(A)}{\ell}\right) \sum_{\gamma \in J[\ell](\mathbb{F}_q)} e^{iA(\gamma)}.$$

Where $\overline{\chi_{\mathrm{Fr}_q}}$ is the largest monic polynomial dividing the characteristic polynomial χ_{Fr_q} of Fr_q that does not have any factors of $(t+1)$ or $(t-1)$.

Thank you!