YY RANTs at YRANT VI:

Pairings arising from Arithmetic Topological Quantum Field Theory

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The Knots and Primes analogy

Topology	Arithmetic
3-Manifold M e.g. S^3	Number Ring $\operatorname{Spec}\mathcal{O}_K$ e.g. $\operatorname{Spec}\mathbb{Z}$
Knot $K : S^1 \hookrightarrow M$	Prime ideal $\mathfrak{p} \triangleleft \mathcal{O}_k$ $\operatorname{Spec}\mathbb{F}_{\mathfrak{p}} \hookrightarrow \operatorname{Spec}\mathcal{O}_K$
Tubular ngbd $V(K)$ of K Torus $\partial V(K)$	p-adic integers $\operatorname{Spec}\mathcal{O}_{K_{\mathfrak{p}}}$ p-adic field $\operatorname{Spec} K_{\mathfrak{p}}$

The Knots and Primes analogy

Theorem

Let $X = \operatorname{Spec} \mathcal{O}_K$ where K is a number field. Then

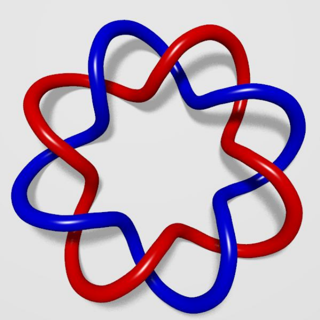
$$\mathrm{inv} : H^3_{\text{ét}}(X, \mathbb{G}_m) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

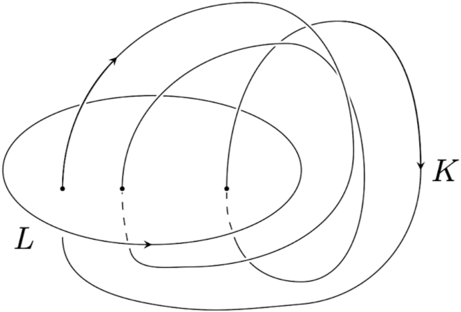
Theorem (Artin-Verdier Duality)

Let $\mathcal{F}$ be a constructible sheaf on X, there is a perfect pairing:

$$H^r(X, F) \times \operatorname{Ext}^{3-r}(F, \mathbb{G}_m) \to H^3(X, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$$

Linking Numbers of Knots





Let K_1, K_2 be knots in a manifold M. One way to compute the linking number involves writing:

$$\mathrm{lk}(K_1, K_2) = \left\langle d^{-1} K_1, K_2 \right\rangle$$

Path Integral Formula

Given 1-forms A_1, A_2 we can also define pairings:

$$(A_1, A_2) := \langle A_1, dA_2 \rangle := \int_M A_1 \wedge dA_2$$

'Theorem' (Path Integral Formula)

Let $\{\xi_i\}_i$ *be a collection of knots. Then:*

$$\int \exp\left(-\pi \langle A, dA \rangle + 2\pi i \sum_i \langle A, \xi_i \rangle\right)$$

$$= \det(\star d)^{-\frac{1}{2}} \cdot \exp\left(-\pi \sum_{i,j} \langle d^{-1}\xi_i, \xi_j \rangle\right)$$

$$= \det(\star d)^{-\frac{1}{2}} \cdot \exp\left(-\pi \sum_{i,j} \mathrm{lk}(\xi_i, \xi_j)\right)$$

The ‘Differential map’

Let K be a number field, $X = \operatorname{Spec} \mathcal{O}_K$. Fix an integer n, assume $\mu_{n^2} \subseteq K$.

What is the arithmetic analogue of the differential map $d : \Omega^1 \to \Omega^2$?

From Artin-Verdier Duality:

$$\langle \cdot, \cdot \rangle : H^1(X, \mathbb{Z}/n\mathbb{Z}) \times \operatorname{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$$

We want a map $d : H^1(X, \mathbb{Z}/n\mathbb{Z}) \to \operatorname{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)$.

There is an isomorphism $\operatorname{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \cong H^1(X, \mathbb{Z}/n\mathbb{Z})^\vee$

The 'Differential map'

We want a map $d : H^1(X, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)$.

Let the *Bockstein map* $\delta : H^1(X, \mathbb{Z}/n\mathbb{Z}) \to H^2(X, \mathbb{Z}/n\mathbb{Z})$ be the connecting homomorphism coming from LES:

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

Then we define:

$$d : H^1(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\delta} H^2(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cup -} H^1(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} \mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)$$

The Abelian CS-Pairing and Linking Numbers

We define the Abelian CS-Pairing to be:

$$(\cdot,\cdot) : H^1(X,\mathbb{Z}/n\mathbb{Z}) \times H^1(X,\mathbb{Z}/n\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

$$(A,B) = \langle A, dB \rangle$$

In fact, $\mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z},\mathbb{G}_m) \cong \mathrm{Cl}(K)/n$, so given ideal classes $[I],[J]$ we can define:

$$\mathrm{lk}_n(I,J) = (d^{-1}[I], d^{-1}[J]) = \left\langle d^{-1}[I], [J] \right\rangle$$

Path Integral Formula for Abelian CS

Theorem (Chung-Kim-Kim-Pappas-Park-Yoo, 2017)

Let $\{I_i\}$ be a set of n-cohomologically trivial ideals, then:

$$\sum_{\rho \in H^1(X, \mathbb{Z}/p\mathbb{Z})} \exp\left[2\pi i \cdot (\rho, \rho) + \sum_i \langle \rho, [I_i] \rangle)\right]$$

$$= p^{(a+b)/2} \cdot \left(\frac{\det \bar{d}}{p}\right) \cdot i^{(a-b)(p-1)^2/4} \exp\left[-2\pi i \cdot \frac{1}{4} \sum_{i,j} \mathrm{lk}_l(I_i, I_j)\right]$$

Remark: AV Duality also holds for Function Fields, and so the CS action and linking pairing can be similarly defined. An analogous version of this theorem holds when X is a curve over a finite field. (C., 2024)

Arithmetic BF Pairing

Now assume K is any number field, S be a finite set of primes, and $U = \operatorname{Spec} \mathcal{O}_K[1/S]$.

There is a similar pairing called the BF-Pairing:

$$BF : H^1(U, \mu_n) \times H^1_c(U, \mathbb{Z}/n\mathbb{Z}) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

$$BF(a, b) = \operatorname{inv}(\delta a \cup b)$$

Where $\delta : H^1(U, \mu_n) \to H^2(U, \mu_n)$ is the bockstein map coming from $0 \to \mu_n \to \mu_{n^2} \to \mu_n \to 0$.

Path Integral Formulas for BF pairing

Theorem (Carlson-Kim, 2020)

Let $U = X = \operatorname{Spec} \mathcal{O}_K$, for K a totally imaginary number field:

$$\sum_{a,b} \exp(2\pi i BF(a,b)) = \left| n \operatorname{Cl}_K[n^2] \right| \left| \mathcal{O}_X^\times / (\mathcal{O}_X^\times)^n \right| \left| \operatorname{Cl}_K / n \right|$$

Ongoing Work

- Can we express these pairings in terms of more ‘classical’ Number Theory pairings, e.g. Hilbert Symbols?

Thank you!