An introduction to the étale fundamental group

GlaMS Examples Seminar

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Review of the topological fundamental group

Definition

Given a topological space X and a basepoint $x_0 \in X$, the *fundamental group* $\pi_1(X, x_0)$ is the group of equivalence classes of loops based at x_0, with group operation given by concatenation.

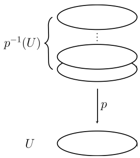
Example

The circle S^1 has fundamental group isomorphic to $\mathbb{Z}$.

Covering Spaces

Definition

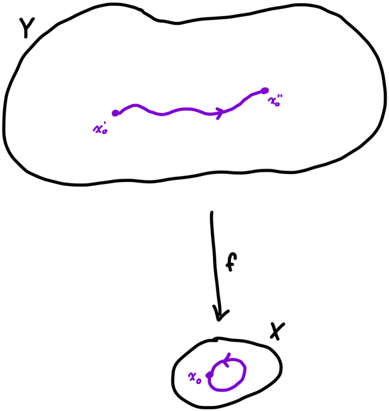
A *covering map* $f : Y \to X$ is a map of topological spaces such that Y is path connected, and each point $x \in X$ has an open neighbourhood $U \ni x$ such that $f^{-1}(U)$ is a disjoint union of open sets, each of which is homeomorphic to U.



If a covering space is in addition simply connected, then it is called the *Universal Cover*, denoted by $\tilde{X}$.

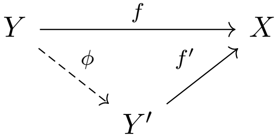
Action of Fundamental Group

Fix a basepoint $x_0 \in X$, then given a covering map $f : Y \to X$, the fundamental group $\pi_1(X, x_0)$ acts on $f^{-1}(x_0)$, the fibre of x_0, by lifting loops based at x_0 to paths between the fibres.



Subcoverings and Covering Space Automorphisms

Call a map $f' : Y' \to X$ a *subcovering* of $f : Y \to X$ if f factors through f'. i.e there exists some $\phi : Y \to Y'$ such that the following diagram commutes:



Denote by $C_X(Y, Y')$ the set of all possible ϕ such that the diagram commutes. If $Y = Y'$, then $C_X(Y, Y)$ forms a group, which we will call the group of covering space automorphisms, $\mathrm{Aut}(Y/X)$. Note that $\mathrm{Aut}(Y/X)$ acts on the fibres of a point $f^{-1}(x_0)$.

Galois Correspondence

Turns out that the group action of $\pi_1(X, x_0)$ and $\mathrm{Aut}(Y/X)$ on the fibres of x_0 are closely related. We will state the following theorem without proof:

Theorem

Let X be a topological space. There is a bijection between the sets of groups:

1. *Quotients of the fundamental group $\pi_1(X, x_0)$.*
2. *The automorphism group $\mathrm{Aut}(Y/X)$ for based covering maps $Y \to X$*

And the groups are isomorphic.

In particular, if $\tilde{X}$ is the universal cover of X, then

$$\pi_1(X, x_0) \cong \mathrm{Aut}(\tilde{X}/X)$$

Problem

We wish to define a similar notion to the fundamental group for algebraic objects (schemes).

However, the Zariski topology on schemes is too coarse for the notion of loops to be useful.

We instead use the correspondence between the topological fundamental group and covering space automorphisms as motivation to define a "fundamental group".

Finite étale maps are the algebraic analogue to covering maps, and we will use them to define the étale fundamental group.

Étale maps

Definition (Residue field)

Given a prime ideal $\mathfrak{p}$ of a ring A, the *residue field* at p is

$$\kappa(\mathfrak{p}) := A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \cong \mathrm{Frac}(A/\mathfrak{p})$$

We can now define the notion of a finite étale map.

Definition (Finite étale morphisms)

Suppose $f : A \to B$ is a homomorphism of rings, it is *finite étale* if:

1. B is flat and finitely generated as an A-module.
2. For every prime ideal $\mathfrak{p}$ of A, we have that the tensor product $B \otimes_A \kappa(\mathfrak{p})$ is isomorphic to a direct product of finite separable extensions of $\kappa(\mathfrak{p})$.

Worked Example 1

Consider the map $\mathbb{C}[t] \to \mathbb{C}[X]$ given by $t \mapsto X^2$. Is this map étale?

Intuitively we can think of this map as the squaring map on the complex plane. This is not a covering map since it is ramified at 0: i.e. the fibre over 0 is a single point, but the fibre over every other point has size two.

Now lets work through the algebra and see if it agrees with the topological intuition.

Worked Example 1: $\mathbb{C}[t] \to \mathbb{C}[X]$ given by $t \mapsto X^2$

The prime ideals of $\mathbb{C}[t]$ are of the form (0) and $(t - \alpha)$ for $\alpha \in \mathbb{C}$.

Let us first consider $\mathfrak{p} = (0)$, then $\kappa(\mathfrak{p}) = \mathbb{C}(t)$ is field of rational functions in t.

We compute:

$$\mathbb{C}[X] \otimes_{\mathbb{C}[t]} \mathbb{C}(t) \cong \mathbb{C}(X) \cong \mathbb{C}(t)[X]/(X^2 - t)$$

This is a degree 2 (separable) extension of $\mathbb{C}(t)$, so we are good.

Worked Example 1: $\mathbb{C}[t] \to \mathbb{C}[X]$ given by $t \mapsto X^2$

If $\mathfrak{p} = (t - \alpha)$, then $\kappa(\mathfrak{p}) = \mathbb{C}[t]/(t-\alpha) \cong \mathbb{C}$. We compute:

$$\mathbb{C}[X] \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/(t-\alpha) \cong \mathbb{C}[X]/(X^2 - \alpha)$$

There are two cases here:

1. If α is non-zero, then $\alpha = \beta^2$ for some $\beta \in \mathbb{C} \setminus \{0\}$, so
$$\begin{aligned} \mathbb{C}[X]/(X^2-\alpha) &\cong \mathbb{C}[X]/(X-\beta) \times \mathbb{C}[X]/(X+\beta) \\ &\cong \mathbb{C} \times \mathbb{C} \end{aligned}$$
2. If $\alpha = 0$, then this is simply equal to $\mathbb{C}[X]/(X^2)$, which has nilpotents, so cannot be written as a product of extensions of $\mathbb{C}$.

Thus, the map is **not** étale, since the condition fails at $\mathfrak{p} = (t)$.

Worked Example 1.5

Can we modify the previous map so that it becomes étale?

If we ‘remove’ the prime (t) somehow, then we can get an étale map.

This is done by localisation.

Consider the map $\mathbb{C}[t, t^{-1}] \to \mathbb{C}[X]_t = \mathbb{C}[X, X^{-1}]$ given by the exact same map $t \mapsto X^2$.

$\mathbb{C}[t, t^{-1}]$ has the exact same prime ideals as $\mathbb{C}[t]$, **except** that (t) is no longer prime since t is now a unit.

We can repeat the previous calculation, and this time this will give an étale map!

Intuitively we can think of this as removing the origin from the squaring map on $\mathbb{C}$, this is a covering map.

Worked Example 2

Consider the inclusion map $\mathbb{Z} \hookrightarrow \mathbb{Z}[i]$. Is this map étale?

No, because of the prime (2).

We can rewrite $\mathbb{Z}[i]$ as $\mathbb{Z}[x]/(x^2+1)$, and $\kappa((2)) \cong \mathbb{F}_2$.

$$\mathbb{Z}[i] \otimes_{\mathbb{Z}} \kappa((2)) \cong \mathbb{Z}[x]/(x^2+1) \otimes_{\mathbb{Z}} \mathbb{F}_2 \cong \mathbb{F}_2[x]/(x^2+1)$$

This has a nilpotent since $(x+1)^2 = 0$, so cannot be written as a product of finite extensions of $\mathbb{F}_2$.

We say that the prime (2) is *ramified* over this extension.

Worked Example 2

In fact, there are no finite étale maps from $\mathbb{Z}$. This follows from the following 2 theorems (which we will not prove).

Theorem

If $\mathbb{Z} \to R$ is flat if and only if it is an injection.

Theorem

Every extension of $\mathbb{Z}$ is ramified over some prime.

Worked Example 2

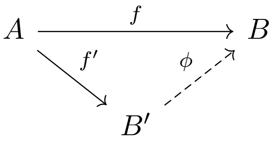
We can play the same trick as last time. If we remove the primes which ramify then we are left with an étale morphism. For example, the map:

$$\mathbb{Z}\left[\frac{1}{2}\right] \hookrightarrow \mathbb{Z}\left[\frac{1}{2}, i\right]$$

is étale.

Subcovers and Automorphisms of étale maps

Suppose $f : A \to B$ is an étale map. We say $f' : A \to B'$ is a subcovering if there is a covering transformation $\phi \in C_A(B', B)$ which is a homomorphism $B' \to B$ such that the following the diagram commutes:



The automorphism group $\mathrm{Aut}(B/A)$ is $C_A(B, B)$.

Basepoints

A basepoint of A, is a morphism $\bar{x} : A \to \Omega$, where Ω is defined to be a separable and algebraically closed field containing the residue field $\kappa(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ in A.

Given an étale cover $f : A \to B$, a fibre of $\bar{x}$ is a morphism $\bar{x}' : B \to \Omega$ such that $\bar{x} = \bar{x}' \circ f$

The group of covering automorphisms $\mathrm{Aut}(B/A)$ acts on the fibres of f by permuting them.

The ‘universal’ pro-étale cover

We want to find some sort of ‘universal’ object so we can define the étale fundamental group as its automorphism group.

Unfortunately this is not possible because no such object exists.

However, we can find a system of finite étale covers $\tilde{A} = (A_i)_{i \in I}$ where $A \to A_i$ is étale, and for any other étale map $A \to B$, the colimit of covering transformations $\varinjlim_{i \in I} C_A(B, A_i)$ gives the fibres of Ω in Y. We call this the *universal pro-étale cover* of A.

The étale fundamental group

Finally, we can define the étale fundamental group:

Definition (Étale fundamental group)

Given a ring A and a basepoint $\bar{x}$, let $\tilde{A} = (A_i)_{i \in I}$ be a universal pro-étale cover of A.
The *étale fundamental group* of A is defined to be the profinite limit of the automorphism groups of $\tilde{A}$.

$$\pi_1(A, \bar{x}) := \varprojlim_{i \in I} \mathrm{Aut}(A_i/A)$$

Worked Example 1

We try to compute the étale fundamental group of $A = \mathbb{C}[t, t^{-1}]$.

From earlier we know that $\mathbb{C}[t, t^{-1}] \to \mathbb{C}[X, X^{-1}]$ given by $t \mapsto X^2$ is finite étale.

The covering automorphisms of this map are given by the identity map and the map $X \mapsto -X$.

Worked Example 1

In fact for $A_n = \mathbb{C}[X, X^{-1}]$, the map $\phi_n : t \mapsto X^n$ is finite étale for all positive integers n.

The associated covering automorphisms in $\mathrm{Aut}(A_n/A)$ are given by $X \mapsto \zeta X$ for ζ an nth root of unity. So the automorphism group is isomorphic to the group of nth roots of unity μ_n:

$$\mathrm{Aut}(A_n/A) \cong \mu_n \cong \mathbb{Z}/n\mathbb{Z}$$

This forms a universal pro-cover, so the étale fundamental group is:

$$\pi_1(\mathbb{C}[t, t^{-1}]) \cong \varprojlim_n \mathbb{Z}/n\mathbb{Z} \cong \hat{\mathbb{Z}}$$

Worked Example 2

We've seen before that $\mathbb{Z}$ has no étale covers, so it has trivial fundamental group.

However, if we localise at a prime and look at $\mathbb{Z}\left[\frac{1}{p}\right]$, then every étale map corresponds to an extension of $\mathbb{Z}$ that ramifies only at p.

These have automorphism group isomorphic to $\mathrm{Gal}(K/\mathbb{Q})$, where K and $\mathbb{Q}$ are the fields of fractions of our rings.

Thus the étale fundamental group of $\mathbb{Z}\left[\frac{1}{p}\right]$ is the inverse limit of all Galois groups unramified outside p. i.e. it is the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside p

$$\pi_1\left(\mathbb{Z}\left[\frac{1}{p}\right]\right) \cong \mathrm{Gal}(\mathbb{Q}_{\{p\}}/\mathbb{Q})$$