Key
- To write
- To say
- Supplementary Notes

2023/03/01 Talk Plan

Agenda

① Correspondence between Knots & Primes
② Decomposition of Knots & Primes
③ TQFTs as a method to generate topological invariants
④ 1D TQFTs
⑤ Towards Arithmetic TQFTs

§1.1: Knots

M: Compact 3 manifold, e.g S^3
K: Knot, i.e an embedding $S^1 \hookrightarrow M$.

One invariant of knots is the knot group, the fundamental group of the knot complement

$G_K = \pi_1(M \setminus K)$

$M \setminus K$ not compact, instead we can remove a tubular neighbourhood of M.

V_K: Tubular ngbd of K.

$M_K := M \setminus \mathrm{int}(V_K)$

$L = K_1 \amalg K_2 \amalg \cdots \amalg K_r$ Link

$G_L := \pi_1(M \setminus L)$ Link group

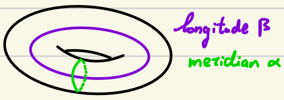
We consider the following fundamental groups

$\pi_1(M)$ depends on M, but $\pi_1(S^3) = 0$

$\pi_1(K) \cong \pi_1(S^1) \cong \mathbb{Z}$

$\pi_1(V_K) \cong \pi_1(K) \cong \mathbb{Z}$

$\pi_1(\partial V_K) \cong \pi_1(S^1 \times S^1) \cong \mathbb{Z}^2 = \langle \alpha, \beta \mid [\alpha, \beta] = 1 \rangle$



§1.2 Primes

K: Number field (Usually $\mathbb{Q}$ for most of this talk)

$\mathcal{O}_K$: Ring of integers of K ($\mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$)

p: Prime ideal of $\mathcal{O}_K$.

Given a prime ideal p, there are 2 constructions we can do:

K_p: p-adic completion of K.

$\mathbb{F}_p$: $\mathcal{O}_K/p$, the residue field at p.

Will define ramification in the next section, but just stating results for now.

Big unsolved problem: To understand the structure of $\mathrm{Gal}(\bar{K}/K)$ Every finite extension is ramified at finitely many places, so suffices to study $G_S(K)$

Let $S = \{p_1, \cdots, p_r\}$, $G_S(K) := \mathrm{Gal}(K_S/K)$, K_S = maximal ext of K unramified outside of S.

Still very hard to compute in general, don't even know if f.g. for a single prime.

There is an algebraic analogue of the fundamental group called the étale fundamental group: étale maps are flat and unramified which morally corresponds to a covering map of topological spaces, so the étale fundamental group is defined as a limit of the automorphism groups of all finite étale maps.

$\pi_1(\mathrm{Spec}\, \mathcal{O}_K) \cong Gal(K^{ur}/K)$, in particular $\pi_1(\mathrm{Spec}\,\mathbb{Z}) = 1$

$\pi_1(\mathrm{Spec}\,\mathcal{O}_K \setminus S) = G_S(K)$ Galois group of maximal extension unramified outside S.

$\pi_1(\mathrm{Spec}\,\mathbb{F}_p) \cong Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$ Generated by Frobenius.

$\pi_1(\mathrm{Spec}\,\mathcal{O}_{K_p}) \cong Gal(K_p^{ur}/K_p) \cong Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$

$\pi_1(\mathrm{Spec}\,K_p) \cong Gal(\overline{K}_p/K_p)$ - hard to calculate, instead look at "tame quotient"

$\pi_1^t(\mathrm{Spec}\,K_p) \cong \langle \sigma, \tau \mid \tau^{p-1}[\tau,\sigma]=1\rangle$ Profinite group

π_1^t: tame fundamental group, take tamely ramified extensions (ramification degree at each $\mathfrak{p}$ is coprime to char $k(\mathfrak{p})$)

$\exists$ surjection $\pi_1(x) \twoheadrightarrow \pi_1^t(x)$

lift of Frobenius map

σ: (lift of) Frobenius, $\sigma(x) \equiv x^p \bmod p$. Explicitly $\sigma(\zeta_n) = \zeta_n^p$

τ: Monodromy - generator of the inertia subgroup $I_{K_p} = \mathrm{Ker}(\mathrm{res}: Gal(\overline{K}_p/K_p) \longrightarrow Gal(\overline{\mathbb{F}}_p/\mathbb{F}))$

Explicitly: $\tau(\zeta_n) = \zeta_n$, $\tau(\sqrt[n]{p}) = \zeta_n \sqrt[n]{p}$

Generator of I_{K_p} exists because we are taking tame quotient (Tame Quotient $= G_0/G_1$ is a subgroup of $k_i^\times$ which is cyclic)

So we have the following correspondence:

Knots	Primes
3-Manifold M	Number Ring $\mathcal{O}_K$
Knot $S^1 \hookrightarrow M$	Prime ideal $\mathrm{Spec}\,\mathbb{F}_p \hookrightarrow \mathrm{Spec}\,\mathcal{O}_K$
Tubular ngbd V_K	p-adic number ring $\mathcal{O}_{K_p}$
Boundary of ngbd ∂V_K	p-adic local field K_p
Longitude β	Frobenius σ
Meridian α	Monodromy τ
Link Group $\pi_1(M \setminus L)$	Galois Group w/ restricted ramification $\pi_1(\mathrm{Spec}\,\mathcal{O}_K \setminus S) \cong G_S(K)$

§2 Decomposition of Knots and Primes

§2.1 Knots

Defn (Ramified Covering Space)

Let $L = L_1 \amalg L_2 \amalg \cdots \amalg L_s \hookrightarrow M$ be a link, a cts map $f: N \to M$ is a ramified covering over L if:

- $f|_{N \setminus f^{-1}(L)}: N \setminus f^{-1}(L) \longrightarrow M \setminus L$ is a covering map
- For each $y \in f^{-1}(L)$, $\exists$ ngbds $D^2 \times I \cong U \ni y$, $D^2 \times I \cong V \ni f(y)$, s.t. $f_*: U \to V$ is $(z \mapsto z^e) \times id$ (identifying $D^2 \cong \{|z| \le 1\} \in \mathbb{C}$

This is like ramification of Riemann Surfaces, have ramification over a co-dimension 2 submanifold so intuitively the map is "wrapping" around the submanifold

e is fixed for each component link K_i, we call this the ramification degree of K_i

Let $X := M \setminus L$, $Y := N \setminus f^{-1}(L)$, let $G := Gal(M/S^3) = Gal(Y/X)$ i.e this is the Quotient of $\pi_1(S^3 \setminus L)$ that corresponds to the covering space $N \setminus f^{-1}(L)$

Let K be a knot in M that is either a component of L or disjoint to L. V_K tubular ngbd

$f^{-1}(K)$ will be a link in N, say $f^{-1}(K) = K_1 \cup \cdots \cup K_r$, V_{K_i}: the component of $f^{-1}(V_K)$ containing K_i

Pick a basepoint $x \in \partial V_K$, then G acts on $f^{-1}(x) = \{y_1, \cdots, y_n\}$, where $n = \#G$. $x \notin L$

G acts transitively on $f^{-1}(x) \Rightarrow G$ acts transitively on $\partial K_1, \cdots, \partial K_r$ and thus $K_1, \cdots, K_r$

We define the stabiliser of K_i to be the decomposition group

Decomposition Group: $D_{K_i} := \{g \in G \mid g(K_i) = K_i\}$

The decomposition groups are all conjugate to each other:

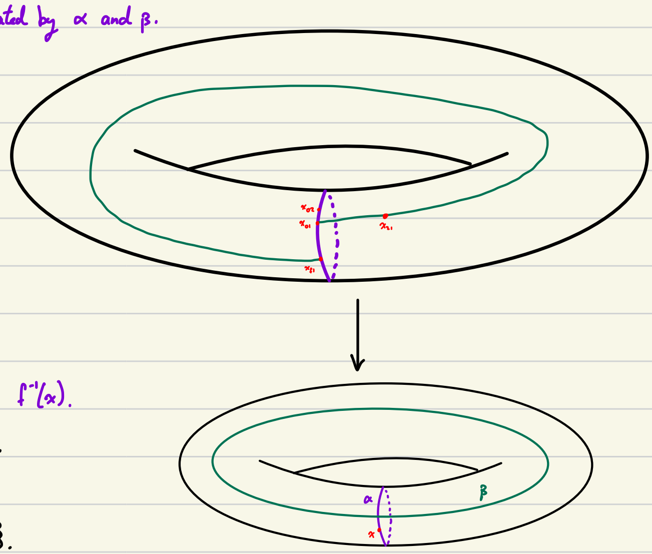
If $g(K_i) = K_j$, then $D_{K_j} = g D_{K_i} g^{-1}$.

Moreover, g induces a homeomorphism of tubular boundaries

$$g|_{\partial V_{K_i}} : \partial V_{K_i} \xrightarrow{\sim} \partial V_{K_j}.$$

In particular for $g \in D_{K_i}$, $g \mapsto g|_{\partial V_{K_i}}$ gives a covering space automorphism of ∂V_{K_i} and induces an isomorphism:

$$D_{K_i} \cong Gal(\partial K_i / \partial K).$$ This is a cover of a torus, is generated by α and β.



The map $g \mapsto g|_{K_i}$ induces a homomorphism:

$$D_{K_i} \longrightarrow Gal(K_i/K)$$

Define the inertia group I_{K_i} to be the kernel of this homomorphism.

Meridians get sent to the identity, so I_{K_i} is generated by α.

Again I_{K_i} & I_{K_j} are conjugate.

Upshot: We can understand D_{K_i} and I_{K_i} by looking at how they act on $f^{-1}(x)$.

Fix x^i_{01} a point in $f^{-1}(x) \cap \partial V_{K_i}$. Let the orbit of x^i_{01} under α be $\{x_{01}, x_{02}, \cdots, x_{0e}\}$.

Then $\# I_{K_i} = e$, the ramification degree.

Define $x_{mn} = \beta^m \cdot x_{0n}$, and define f to be the minimal m s.t. $x_{f1} \in \{x_{01}, \cdots, x_{0e}\}$.

f is the <u>covering degree</u> of K_i over K.

$$ef = |f^{-1}(x) \cap \partial K_i|$$

Then we have: $efr = n = \# G$.

Some special cases:

$D_{K_i} = 1 \iff e = f = 1,\ r = n$ K decomposes completely.

$D_{K_i} = G \iff ef = n,\ r = 1$

$I_{K_i} = 1 \iff e = 1,\ fr = n$

$I_{K_i} = G \iff e = n,\ f = r = 1$ K totally ramified

§2.2 Primes

Defn (Ramification of a prime)

Let L/K be a finite extension of number fields. p a prime ideal of $\mathcal{O}_K$.

$p\mathcal{O}_L$ is an ideal in $\mathcal{O}_L$, but not necessarily prime.

$\mathcal{O}_L$ Dedekind domain so has unique prime factorisation

$$p\mathcal{O}_L = P_1^{e_1}\cdots P_r^{e_r} \qquad P_i \cap \mathcal{O}_K = p$$

p is unramified in L if $e_i = 1 \ \forall i$. e_i is the ramification degree of P_i

Quotienting induces an extension of residue fields: Let $f_i = [\mathcal{O}_L/P_i : \mathcal{O}_K/p]$, then if $\mathcal{O}_K/p = \mathbb{F}_p$, $\mathcal{O}_L/P_i = \mathbb{F}_{p^f}$.

f_i is called the residue degree of P_i

Theorem: $\sum e_i f_i = [L:K] = n$ Proof omitted, quite involved.

Suppose now that L/K is Galois, then $Gal(L/K)$ acts transitively on $\{P_1, \dots, P_r\}$ Pf omitted

Then for $\sigma \in Gal(L/K)$:

$$p\mathcal{O}_L = \sigma(p)\mathcal{O}_L = \sigma(P_1)^{e_1}\sigma(P_2)^{e_2}\cdots\sigma(P_r)^{e_r} \quad \Rightarrow e_1 = e_2 = \dots = e_r = e$$
$$\Rightarrow \mathcal{O}_L/P_i \cong \mathcal{O}_L/\sigma(P_i)$$
$$\Rightarrow f_1 = f_2 = \dots = f_r = f$$
$$\Rightarrow efr = n$$

Decomposition group: Stabiliser of P_i

$$D_{P_i} = \{\sigma \in Gal(L/K) \mid \sigma(P_i) = P_i\}$$ Note again that the decomposition groups are conjugate.

By orbit-stabiliser: D_{P_i} has order ef.

Fixing a P_i, $D_{P_i} \cong Gal(L_{P_i}/K_p)$.

Moreover: $\sigma \longmapsto (\bar{\sigma} : \alpha \bmod P_i \longmapsto \sigma(\alpha) \bmod P_i)$ defines a surjective map: Surjectivity not obvious, but proof omitted.

$$D_{P_i} \cong Gal(L_{P_i}/K_p) \longrightarrow Gal(\mathbb{F}_{p^f}/\mathbb{F}_p)$$ cyclic of order f, generated by frobenius.

We define I_{P_i} to be the kernel of this map. Then $|I_{P_i}| = e$ Inertia groups are conjugate.

Special Cases:

$D_{P_i} = 1 \iff e = f = 1, \ r = n$ $\quad p$ decomposes completely.

$D_{P_i} = G \iff ef = n, \ r = 1$

$I_{P_i} = 1 \iff e = 1, \ fr = n$

$I_{P_i} = G \iff e = n, \ f = r = 1$ $\quad p$ totally ramified

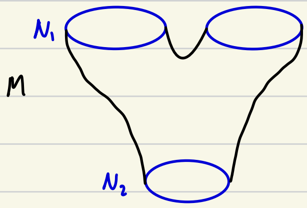
Extra complexity in prime case: D_{x_i} is quotient of $\mathbb{Z}^2$ so always abelian, but D_{P_i} is generally non-abelian

§3. TQFTs

Defn (Category of Bordisms)

$Bord_n$ Objects: $n-1$ manifolds w/ choice of orientation (including empty manifold)

Morphisms: Given objects N_1, N_2, a morphism $M \in Hom(N_1, N_2)$ is a n-fold s.t $\partial M = \bar{N}_1 \sqcup N_2$, up to orientation preserving diffeomorphisms.

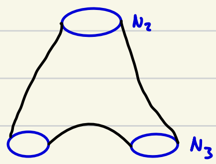


Orientation needed so we can define an "IN" vs. "OUT"

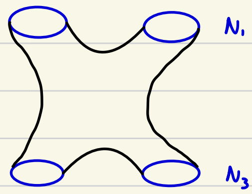
In physics, can view this orientation as "time evolution" of a state

Composition of morphisms: Given $M \in Hom(N_1, N_2)$, $M' \in Hom(N_2, N_3)$, we define the composition $M' \circ M$ to be the n-fold obtained by gluing M and M' along N_2

e.g: $M' =$



then $M' \circ M =$



Identity: $Id: N \longrightarrow N$ is given by $N \times [0,1]$

Intuitively, this means you can "multiply" objects in the category

Note that $Bord_n$ is monoidal with operation disjoint union. The unit object is $\emptyset$.

Defn (TQFT)

An n-dimensional Topological Quantum Field Theory is a symmetric monoidal functor:

$$\tau : Bord_n \longrightarrow Vect_k$$

Explicitly is the following set of data:

- For an $n-1$ manifold N, $\tau(N)$ is a k-Vector space
- For an n manifold $M \in Hom(N_1, N_2)$, $\tau(M)$ is a linear map:
$$\tau(M) : \tau(N_1) \longrightarrow \tau(N_2)$$

These come from the defn of a functor

- $\tau(N_1 \sqcup N_2) \cong \tau(N_1) \otimes \tau(N_2)$
- $\tau(\emptyset) \cong k$
- $\tau(M_1 \sqcup M_2) \cong \tau(M_1) \otimes \tau(M_2)$

"Monoidal properties"

- $\tau(N_1) \otimes H(N_2) \cong \tau(N_2) \otimes \tau(N_1)$

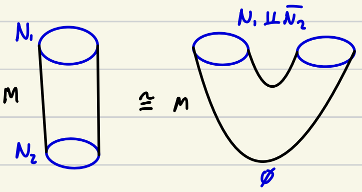
Symmetry

Thm $\tau(\bar{N}) \cong \tau(N)^*$ Proof omitted

Idea: We can "forget" the time evolution intuition from physics because:

Finite dimensionality?

Is a corollary of the proof that $\tau(\bar{N}) \cong \tau(N)^*$



So can think of $\tau(M) \in Hom(\tau(N_1), \tau(N_2))$

as instead $\tau(M) \in Hom(\tau(N_1 \sqcup \bar{N}_2), k) \cong (\tau(N_1) \otimes \tau(N_2)^*)^* \cong \tau(N_1)^* \otimes \tau(N_2)$

Algebraically: $\tau(M) \in Hom(\tau(N_1), \tau(N_2)) \cong \tau(N_1)^* \otimes \tau(N_2) \cong \tau(\bar{N}_1) \otimes \tau(N_2) \cong \tau(\bar{N}_1 \sqcup N_2) \cong \tau(\partial M)$

This means we can associate to each n-manifold a vector instead of a linear map.

In particular, if M is a manifold w/o boundary, then

$\tau(M) \in \tau(\emptyset) \cong k$ is just a number

This is diffeomorphism-invariant, so TQFTs can be thought of as a way to generate invariants on manifolds

§4. 1D TQFTs

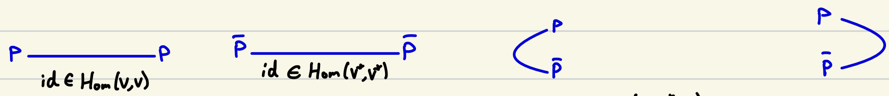
The only connected 0-manifold is the point P and $\bar{P}$ pt w/ reversed orientation

The only connected 1-manifolds are I and S^1.

Let $V = \tau(P)$, $V^* = \tau(\bar{P})$.

I is the identity morphism between P and P, so $\tau(I)$ must be identity map

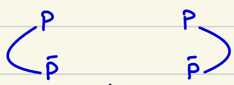
$\tau(I) = id \in Hom(V,V) \cong Hom(\bar{V},\bar{V}) \cong V^* \otimes V$



$id \in Hom(V,V)$ $id \in Hom(V^*,V^*)$ $\sum e_i^* \otimes e_i \in V^* \otimes V \cong Hom(k, V^* \otimes V)$ $(\sigma \otimes v \mapsto \sigma(v)) \in Hom(V^* \otimes V \mapsto k)$

What about $\tau(S^1)$?

Have composition of maps:



$$\tau(S^1): \mathbb{C} \longrightarrow V^* \otimes V \longrightarrow \mathbb{C}$$

$\therefore$ $\tau(S^1) = \dim(V)$

All morphisms in $Bord_1$ are made up of the above, so every 1D TQFT is uniquely determined by $\tau(P) = V$.

In fact: • Every 2D TQFT corresponds to a Frobenius algebra and vice versa

• 3D TQFTs harder, but related to Hopf algebras.

§5 Towards Arithmetic TQFTs

From our analogy earlier, we can think of $\mathcal{O}_K$ as a 3-fold, and local field K_p as a tubular boundary of the knot (2-fold)

If S is a set of primes, we can think of $(\operatorname{Spec} \mathcal{O}_K) \setminus S$ as a 3-manifold, with "boundary" being the local fields $\coprod_{p \in S} \operatorname{Spec}(K_p)$, which can be thought of as a 2-manifold.

$\therefore$ We can think of $(\operatorname{Spec} \mathcal{O}_K) \setminus S$ as a 3-Bordism between the local fields at primes in S.

How do we think about orientation in this setting? I'm not actually sure.

Under this setting, an arithmetic TQFT should assign:

$\tau : \operatorname{Spec} \mathcal{O}_K \longmapsto$ A number Since $\mathcal{O}_K$ thought of as a manifold w/o boundary.

$\tau : \operatorname{Spec} K_p \longmapsto$ A vector space

$\tau : (\operatorname{Spec} \mathcal{O}_K) \setminus S \longmapsto$ A vector in $\bigotimes_{p \in S} \tau(\operatorname{Spec} K_p)$