Arithmetic triple linking numbers

YRANT V - Cambridge

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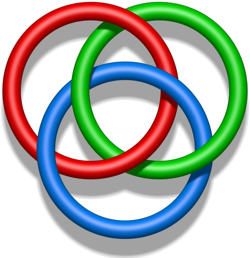
The Knots and Primes analogy

In the 1960s, Barry Mazur noticed an analogy between knots in a three manifold and primes in a number field.

Topology	Arithmetic
3-Manifold M e.g. S^3	Number Ring $\operatorname{Spec}\mathcal{O}_K$ e.g. $\operatorname{Spec}\mathbb{Z}$
Knot $K : S^1 \hookrightarrow M$	Prime ideal $\mathfrak{p} \triangleleft \mathcal{O}_k$ $\operatorname{Spec}\mathbb{F}_{\mathfrak{p}} \hookrightarrow \operatorname{Spec}\mathcal{O}_K$
Tubular ngbd $V(K)$ of K Torus $\partial V(K)$	p-adic integers $\operatorname{Spec}\mathcal{O}_{K_{\mathfrak{p}}}$ p-adic field $\operatorname{Spec} K_{\mathfrak{p}}$
Linking Number $\mathrm{lk}(L, K)$	Power Residue Symbol $\left(\frac{\mathfrak{p}}{\mathfrak{q}}\right)_n$

Borromean Rings

The Borromean Rings is a set of three rings that are pairwise unlinked, but altogether linked.



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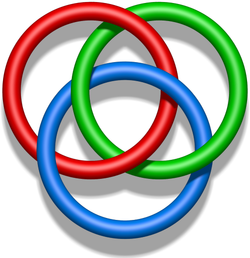

The Borromean Rings cannot be detected by linking numbers, but the *Milnor invariants* can be defined for multiple knots and can detect phenomena such as the Borromean Rings.

Arithmetic Milnor Invariants

We wish to define multiple linking numbers mod l. Fix a prime number l, and suppose $S = \{p_1, p_2, p_3\}$ is a set of primes of $\mathcal{O}_K$ with $p_i \equiv 1 \pmod{l}$.

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Let τ_i be a monodromy over p_i and σ_i be a lift of the p_i Frobenius. These are elements of $G_S = \mathrm{Gal}(K_S/K)$ which satisfy the following:

$$\tau_i(\zeta_{l^n}) = \zeta_{l^n}; \quad \tau_i(\sqrt[l^n]{p_i}) = \zeta_{l_n} \sqrt[l^n]{p_i}$$
$$\sigma_i(\zeta_{l^n}) = \zeta_{l^n}^{p_i}; \quad \sigma_i(\sqrt[l^n]{p_i}) = \sqrt[l^n]{p_i}$$

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If $K = \mathbb{Q}$ there is a theorem of Koch ([KGR70]) where the maximal pro-l quotient of the Galois Group G_S has the following presentation:

$$G_S(l) \cong \left\langle x_1, \ldots x_r | x_1^{p_1 - 1}[x_1, y_1] = \cdots = x_r^{p_r - 1}[x_r, y_r] = 1 \right\rangle$$

Where x_i represents the monodromy τ_i, and y_i are words that represent σ_i.

Arithmetic Milnor Invariants

Suppose these primes are pairwise unlinked mod l. i.e. $\left(\frac{p_i}{p_j}\right)_l = 1$

There is a surjective homomorphism ψ from G_S to the Heisenberg Group with entries in $\mathbb{Z}/l\mathbb{Z}$ such that:

$$\tau_1 \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; \tau_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

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Moreover, the Frobenius σ_3 of p_3 will be mapped to a matrix of the form:

$$\begin{pmatrix} 1 & 0 & \mu \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Arithmetic Triple Linking Numbers

Let L/K be the extension of K corresponding to the kernel $\ker\psi$. Then $\mathrm{Gal}(L/K) \cong H_3(\mathbb{Z}/l\mathbb{Z})$ is ramified only at the primes p_1, p_2, and also dependent only on p_1, p_2.

Arithmetic Triple Linking Numbers

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If K contains a primitive lth root of unity ζ_l, then we define the mod l linking number of p_1, p_2, p_3 to be:

$$[p_1, p_2, p_3]_l = \zeta_l^{\mu}$$

The triple linking number measures whether p_3 splits or is inert in L/K.

Rédei Triple Symbols

In the case of $l = 2$, Rédei (1939) gave an explicit construction of the extension $L/\mathbb{Q}$ and interprets the Rédei Triple Symbol $[p_1, p_2, p_3]$ as a generalisation of the Legendre Symbol. [Réd39]

[1] Amano-Kodani-Morishita-Sakamoto-Ogasawara-Yoshida

Rédei Triple Symbols

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In 2013, a paper by Amano et al.[1] gives a way to calculate Rédei Triple Symbols through counting lattice points on quadratic forms, and can also be expressed as a Fourier coefficient of a modular form of weight one, gives an explicit and constructive example of the theorem by Weil-Langlands and Deligne-Serre. [AKMOSY13]

[1] Amano-Kodani-Morishita-Sakamoto-Ogasawara-Yoshida

Outline of Proof

1. Construct a $\mathbb{C}$-representation $\rho : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ of the Rédei extension $L/\mathbb{Q}$ such that

$$[-p_1, p_2, p_3] = \frac{1}{2} \operatorname{tr} \rho(\sigma_{p_3})$$

Then the triple symbols are encoded in the coefficients of the Artin L-function:

$$L(\rho, s) = \prod_p L_p(\rho, s)$$

$$L_p(\rho, s) = \det_{V^{I_p}} \left(I - t\rho(\sigma_p)\right)^{-1}$$

Outline of Proof

2. Construct a 1-dimensional character $\chi : \mathrm{Gal}(L/M) \to \mathbb{C}^\times$ of a sub-extension L/M such that $\mathrm{Ind}\,\chi$ is equivalent to ρ. Then we have an equality of L-functions:

$$L(\chi, s) = L(\rho, s)$$

In particular L/M is abelian so we can factor χ through the class group H_M to get a Hecke character and Hecke L-function that is also equal.

$$\boldsymbol{\chi} = \chi \circ \left(\frac{L/M}{\cdot}\right) : H_M \to \mathbb{C}^\times$$

$$L(\boldsymbol{\chi}, s) = L(\chi, s) = L(\rho, s)$$

Outline of Proof

3. $M = \mathbb{Q}(\sqrt{-p_1p_2})$ is a quadratic extension over $\mathbb{Q}$. Since there is a bijection between the class group H_M and quadratic forms with the same discriminant, let Q_i be the quadratic form associated to ideal class C_i.

Lemma

Let:

$$a(C_i, n) := \#\{(x, y) \in \mathbb{Z}^2 | Q_i(x, y) = n\}$$

$$b(C_i, n) := \#\{\mathfrak{a} \in C_i^{-1} : N\mathfrak{a} = n\}$$

Then

$$a(C_i, n) = 2b(C_i, n)$$

Outline of Proof

4. We can now rewrite the L function as follows

$$\begin{aligned} L(\chi, s) &= \sum_{\mathfrak{a} \triangleleft \mathcal{O}_M} \chi([\mathfrak{a}]) N\mathfrak{a}^{-s} \\ &= \sum_{i=0}^{h_M - 1} \chi(C_i^{-1}) \sum_{\mathfrak{a} \in C_i^{-1}} N\mathfrak{a}^{-s} \\ &= \sum_{i=0}^{h_M - 1} \chi(C_i^{-1}) \sum_{n=1}^{\infty} b(C_i, n) n^{-s} \\ &= \frac{1}{2} \sum_{i=0}^{h_M - 1} \chi(C_i^{-1}) \sum_{n=1}^{\infty} a(C_i, n) n^{-s} \end{aligned}$$

So we can recover the values of the Rédei triple symbols by counting lattice points on Quadratic forms Q_i.

Outline of Proof

5. If we define:

$$\begin{cases} \theta(C_i, z) & := \dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty} a(C_i, n) q^n = \dfrac{1}{2} \sum_{(x,y)\in\mathbb{Z}^2} q^{Q_i(x,y)} \\ \Theta(z) & := \displaystyle\sum_{i=0}^{h_M - 1} \boldsymbol{\chi}(C_i)\theta(C_i, z) \end{cases}$$

Then θ is a modular form, and Θ is a cuspidal eigenform. Moreover

$$L(\Theta, s) = L(\boldsymbol{\chi}, s) = L(\rho, s)$$

Giving an explicit and constructive example of the Langlands correspondence.

Question

The Rédei Triple Symbol is the mod 2 triple linking number of primes. Is it possible to extend this result to mod 3 triple linking numbers?

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Since our base field K will need to contain the cubic roots of unity, we will have to work over $K = \mathbb{Q}(\sqrt{-3})$

A paper by Amano, Mizusawa, and Morishita [AMM18] gives an explicit construction of the extension L/K in this case.

Progress so far

Managed to construct analogues for ρ and χ in the mod 3 linking number case.

$$\rho : \mathrm{Gal}(L/K) \cong H_3(\mathbb{Z}/3\mathbb{Z}) \to \mathrm{GL}_3(\mathbb{C})$$

$$\begin{pmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

This map sends $\begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$ to $\begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta \end{pmatrix}$, and so it satisfies:

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 = \frac{1}{3} \operatorname{tr} \rho(\sigma_{\mathfrak{p}_3})$$

Progress so far

Considering $M = L^H$ to be the intermediate field fixed by the subgroup

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ & 1 & a \\ & & 1 \end{pmatrix}; \quad a, b \in \mathbb{F}_3 \right\}$$

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The character χ given by $\chi \begin{pmatrix} 1 & a & b \\ & 1 & a \\ & & 1 \end{pmatrix} = \zeta^b$ satisfies $\rho = \operatorname{Ind} \chi$.

Thus defining $\boldsymbol{\chi} = \chi \circ \left(\frac{L/M}{\cdot} \right) : H_M \to \mathbb{C}^\times$ we once again have the following equalities of L-functions:

$$L(\boldsymbol{\chi}, s) = L(\chi, s) = L(\rho, s)$$

Difficulties

- The intermediate field M is a degree 3 extension of $K = \mathbb{Q}(\sqrt{-3})$ rather than a degree 2 extension of $\mathbb{Q}$, so there isn't any well documented bijection between ideal classes of H_M and binary cubic forms.

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- The proof of the lemma $a(C_i, n) = 2b(C_i, n)$ requires the fact that $\mathcal{O}_M^\times = \{\pm 1\}$, since the $\mathcal{O}_M^\times$ will be infinite in our case, we would have to count equivalence classes of lattice points rather than lattice points.

Thank you!

References

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