On the q-analogue of Kostant's partition function

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This is a collection of miscellaneous findings and observations from my Research Project with Dr. Rong Zhou in the summer of 2021.

1 Preliminaries

1.1 Embedding A_n into $\mathbb{R}^{n+1}$

$\mathbb{R}^{n+1}$ contains an embedding of A_n by taking the roots of the root system to be $\alpha_{ij} = e_i - e_j$ for $(e_i)_{i=0}^n$ the standard basis of $\mathbb{R}^{n+1}$.

We take the positive roots of A_n to be the set $A_n^+ = \{\alpha_{ij} | 0 \leq i < j \leq n\}$. This set of positive root corresponds to a base $\beta_i = \alpha_{i,i+1}$ for $i = 0, 1, \cdots, n-1$.

We define Λ_n to be the root lattice of A_{n-1}, i.e. the vectors in $\mathbb{R}^n$ that are integer linear combinations of the roots. Similarly we let Λ_n^+ to be the positive root lattice, which are the vectors which are positive integer linear combinations of the positive roots.

We additionally define $h(\lambda)$ to be the sum of the coefficients when λ is expressed in terms of the basis. This is the *height* of λ.

1.2 The q-Analogue of Kostant's Partition Function and the alternating Weyl sum

Let $\mathcal{P}(\lambda, q^{-1})$ denote the q-analogue of Kostant's Partition Function. It is the generating function where the coefficient of q^{-k} is equal to the number of ways to express λ as a positive integer linear combination of the positive roots of A_n, such that the sum of the coefficients is k.

$$\mathcal{P}(\lambda, q^{-1}) := \sum_{\substack{n_1, \cdots, n_r \\ n_1\alpha_1 + \cdots n_r\alpha_r = \lambda}} q^{-(n_1 + \cdots + n_r)}$$

Define the (Alternating) Weyl Sum to be:

$$\mathfrak{M}_\lambda^0(q^{-1}) := \sum_{\omega \in S_{n+1}} \varepsilon(\omega)\mathcal{P}(\lambda + \omega\rho - \rho, q^{-1})$$

Here, we take $\rho = \frac{1}{2}\sum_{i<j} \alpha_{ij}$ to be half of the sum of the positive roots, and $\omega \in S_{n+1}$ acts on ρ by permuting the co-ordinates of ρ, and $\varepsilon(\omega)$ gives the sign of the permutation ω.

2 The conjecture

The following conjecture was given by Dr. Rong Zhou, and was the subject of my research project. The original conjecture was given for all root systems, but since my project focused on the case of the root system A_n, I will state the conjecture for A_n.

Conjecture 2.1. *Given* $\mu = (n+1, -1, -1, \cdots, -1)$*, and* s *a positive integer:*

$$\lim_{s\to\infty} q^{\frac{sn(n-1)}{2}} \mathfrak{M}^0_{s\mu}(q^{-1}) = p_n(q)$$

In other words, for sufficiently large s, the alternating Weyl sum multiplied by $q^{\frac{sn(n-1)}{2}}$ is eventually a constant polynomial. Note in particular that the power of q that we multiply by, $\frac{sn(n-1)}{2}$ is the height of the vector $s\mu$.

3 A useful identity

Proposition 3.1. *For any* λ *and* ρ *as above, we have*

$$\begin{aligned}\mathfrak{M}^0_{\lambda}(q^{-1}) &= \sum_{\omega\in S_{n+1}} \varepsilon(\omega)\mathcal{P}(\lambda+\omega\rho-\rho, q^{-1}) \\ &= \sum_{D\subseteq A^+_{n-1}} (-1)^{|D|}\mathcal{P}\left(\lambda - \sum_{\alpha\in D}\alpha, q^{-1}\right)\end{aligned}$$

This can be proved using the Weyl Character Formula.

From now on we will consider the sum over subsets of the positive roots, instead of the sum over the symmetric group.

4 A recursive identity on calculating the Partition Polynomials

Let $\lambda \in \Lambda_n$, we wish to recursively express $\mathcal{P}(\lambda, q^{-1})$ through partition polynomials of elements in Λ_{n-1}. We do so by making the last co-ordinate of λ equal to 0.

Noting that the only positive roots such that the last co-ordinate is non-zero are of the form α_{in} for $i < n$, let c_i denote the coefficient of α_{in}, then we must have $c_1 + c_2 + \cdots + c_{n-1} = k$, where $-k$ is the last co-ordinate of λ.

We sum over all possible values of c_i such that their sum is k, once the c_i's are fixed, we can factor out q^{-k} and consider partitions of $\lambda - \sum c_i\alpha_{in}$ instead to obtain the following

Proposition 4.1.

$$\mathcal{P}(\lambda, q^{-1}) = h^{-k} \sum_{c_1+\cdots+c_{n-1}=k} \mathcal{P}\left(\lambda - \sum c_i\alpha_{in}, q^{-1}\right)$$

Since the the last co-ordinate of $\lambda - \sum c_i\alpha_{in}$ is 0, this implies that the partitions of the above vectors will not use any positive roots of the form α_{in}, or that it will only use positive roots in the set $\{\alpha_{ij} | 1 \leq i < j \leq n-1\}$. By projecting onto $\mathbb{R}^{n-1}$ by removing the last co-ordinate, each of these polynomials is in fact a partition polynomial of a vector in Λ_{n-1}.

5 Rearranging the Weyl Sum

We try to rearrange the Weyl Sum in a way that can utilise the above identity.

5.1 Considering a 'shortened' Weyl Sum

First we consider a 'shortened' sum where we sum over all subsets of roots that have zero in its final co-ordinate. In other words, let $S' := \{\alpha_{ij} | 1 \leq i \leq j \leq n-1\}$. We consider the sum:

$$\sum_{D \subseteq S'} (-1)^{|D|} \mathcal{P}\left(\lambda - \sum_{\alpha \in D} \alpha, q^{-1}\right)$$

Noting that S' does not contain β_n, all of these $\lambda - \sum_{\alpha \in D} \alpha$ have the same final co-ordinate. Let the the common final co-ordinate be $-k$. Then applying the previous identity from section 4:

$$\mathcal{P}\left(\lambda - \sum_{\alpha \in D} \alpha, q^{-1}\right) = q^{-k} \sum_{c_1 + \cdots + c_{n-1} = k} \mathcal{P}\left(\lambda - \sum_{\alpha \in D} \alpha - \sum_i c_i \alpha_{in}, q^{-1}\right)$$

Where the summands of the second sum all have last co-ordinate zero, which means we have effectively reduced all the partition polynomials to ones of lower dimension.

Going back to our 'shortened' sum:

$$\begin{aligned}
\sum_{D \subseteq S'} (-1)^{|D|} \mathcal{P}\left(\lambda - \sum_{\alpha \in D} \alpha, q^{-1}\right) &= q^{-k} \sum_{D \subseteq S'} \sum_{c_1 + \cdots + c_{n-1} = k} (-1)^{|D|} \mathcal{P}\left(\lambda - \sum_{\alpha \in D} \alpha - \sum_i c_i \alpha_{in}, q^{-1}\right) \\
&= q^{-k} \sum_{c_1 + \cdots + c_{n-1} = k} \sum_{D \subseteq S'} (-1)^{|D|} \mathcal{P}\left(\lambda - \sum_{\alpha \in D} \alpha - \sum_i c_i \alpha_{in}, q^{-1}\right)
\end{aligned}$$

By reversing the order of summation, the term $\sum_{D \subseteq S'} (-1)^{|D|} \mathcal{P}\left(\lambda - \sum_{\alpha \in D} \alpha - \sum_i c_i \alpha_{in}, q^{-1}\right)$ is in fact equal to a Weyl sum from a dimension lower, $\mathfrak{M}^0_{\pi(\lambda - \sum_i c_i \alpha_{in})}(q^{-1})$, where π denotes the projection map by removing the last co-ordinate.

Thus:

$$\sum_{D \subseteq S'} (-1)^{|D|} \mathcal{P}\left(\lambda - \sum_{\alpha \in D} \alpha, q^{-1}\right) = q^{-k} \sum_{c_1 + \cdots + c_{n-1} = k} \mathfrak{M}^0_{\pi(\lambda - \sum_i c_i \alpha_{in})}(q^{-1})$$

5.2 Rearranging the full Weyl sum

We now consider the full Weyl sum over subsets of $S := \{\alpha_{ij} | 1 \leq i \leq j \leq n\}$. Any subset of S can be expressed uniquely as the union of disjoint subsets $D \subseteq S'$ and $T \subseteq S \setminus S'$, thus instead of summing over S, we can sum over S' and $S \setminus S'$ simultaneously.

$$\begin{aligned}
\mathfrak{M}^0_\lambda(q^{-1}) &= \sum_{D \subseteq S} (-1)^{|D|} \mathcal{P}\left(\lambda - \sum_{\alpha \in D} \alpha, q^{-1}\right) \\
&= \sum_{D \subseteq S'} \sum_{T \subseteq S \setminus S'} (-1)^{|D \cup T|} \mathcal{P}\left(\lambda - \sum_{\alpha \in D \cup T} \alpha, q^{-1}\right) \\
&= \sum_{T \subseteq S \setminus S'} (-1)^{|T|} \sum_{D \subseteq S'} (-1)^{|D|} \mathcal{P}\left(\lambda - \sum_{\alpha \in T} \alpha - \sum_{\alpha \in D} \alpha, q^{-1}\right)
\end{aligned}$$

The second summand is precisely the 'shortened' Weyl sum from the previous section. Combining this with the result above, the sum is then equal to:

$$= \sum_{T \subseteq S \setminus S'} (-1)^{|T|} q^{-\kappa(\lambda - \sum_{\alpha \in T} \alpha)} \sum_{c_1 + \cdots + c_{n-1} = \kappa(\lambda - \sum_{\alpha \in T} \alpha)} \mathfrak{M}^0_{\pi(\lambda - \sum_{\alpha \in T} \alpha - \sum_i c_i \alpha_{in})}(q^{-1})$$

Where we define $\kappa(\lambda)$ to be the negative of the last co-ordinate of λ.

This sum may seem like a mess currently, but can be simplified immensely if the conjecture in section 7 is true.

6 Case Study of A_2

To motivate the conjecture of section 7, we first consider a case study of the $A_2 \subset \mathbb{R}^3$ case.

Let $\lambda = (a + b, -a, -b)$, a general vector lying on the positive root lattice of A_2. We can compute directly that $\mathcal{P}(\lambda, q^{-1}) = q^{-(a+b)} + \cdots + q^{-(a+2b)}$.

Then, we note that $\lambda - \alpha_{12} = (a + b - 1, -a + 1, -b)$, and so $\mathcal{P}(\lambda - \alpha_{12}, q^{-1}) = q^{-(a+b-1)} + \cdots + q^{-(a+2b-1)}$.

As a result, $\mathcal{P}(\lambda, q^{-1}) - \mathcal{P}(\lambda - \alpha_{12}, q^{-1}) = q^{-(a+2b)} - q^{-(a+b-1)}$.

We note that λ has height $h(\lambda) = a + 2b$, and in particular, this implies that $q^{h(\lambda)}(\mathcal{P}(\lambda, q^{-1}) - \mathcal{P}(\lambda - \alpha_{12}, q^{-1})) = 1 - q^{b-1}$. An interesting observation is that the final expression is not depending on a.

Now we group the Weyl sum in the way outlined in 5.2:

$$\begin{aligned}
\mathfrak{M}^0_\lambda(q^{-1}) =& [\mathcal{P}(\lambda, q^{-1}) - \mathcal{P}(\lambda - \alpha_{12}, q^{-1})] - [\mathcal{P}(\lambda - \alpha_{13}, q^{-1}) - \mathcal{P}(\lambda - \alpha_{13} - \alpha_{12}, q^{-1})] \\
& - [\mathcal{P}(\lambda - \alpha_{23}, q^{-1}) - \mathcal{P}(\lambda - \alpha_{23} - \alpha_{12}, q^{-1})] + [\mathcal{P}(\lambda - \alpha_{13} - \alpha_{23}, q^{-1}) - \mathcal{P}(\lambda - \alpha_{13} - \alpha_{23} - \alpha_{12}, q^{-1})] \\
=& q^{-h(\lambda)}(1 - q^{b-1}) - q^{-h(\lambda - \alpha_{13})}(1 - q^{b-2}) - q^{-h(\lambda - \alpha_{23})}(1 - q^{b-2}) + q^{-h(\lambda - \alpha_{13} - \alpha_{23})}(1 - q^{b-3}) \\
=& q^{-h(\lambda)}[(1 - q^{b-1}) - (q^2 - q^b) - (q - q^{b-1}) + (q^3 - q^b)] \\
=& q^{-h(\lambda)}(1 - q - q^2 + q^3)
\end{aligned}$$

This in particular implies that $q^{h(\lambda)}\mathfrak{M}^0_\lambda(q^{-1}) = 1 - q - q^2 + q^3$ is a fixed polynomial. Interestingly, λ need not be of the form $s\mu$. The only assumption made on λ is the fact that λ minus any subset of the positive roots, is still in the positive root lattice.

7 A Stronger Conjecture

From the case study in the previous suggestion, it seems that the conjecture can be strengthened as follows: For any λ such that $\lambda - \sum_{\alpha\in\Phi_n^+} \alpha$ is in the positive root lattice, $q^{h(\lambda)}\mathfrak{M}_\lambda^0(q^{-1})$ is a fixed polynomial.

For $n = 3$ this hypothesis is certainly true, and in the next section we aim to use induction to prove this for all n.

8 Implications of the Stronger Conjecture

Suppose the stronger conjecture is true for n, this means that $q^{h(\lambda)}\mathfrak{M}_\lambda^0(q^{-1}) = p_n(q)$ for some polynomial p_n. Alternatively, $\mathfrak{M}_\lambda^0(q^{-1}) = q^{-h(\lambda)}p_n(q)$.

Now suppose $\lambda \in \mathbb{R}^{n+1}$ is a vector in the positive root lattice for $n+1$, and that $\lambda - \sum_{\alpha\in\Phi_n^+} \alpha$ is in the positive root lattice.

Substituting into the expression at the end of section 5.1, we obtain:

$$\begin{aligned}\sum_{D\subseteq S'} (-1)^{|D|}\mathcal{P}\left(\lambda - \sum_{\alpha\in D} \alpha, q^{-1}\right) &= q^{-k} \sum_{c_1+\cdots+c_{n-1}=k} \mathfrak{M}^0_{\pi(\lambda-\sum_i c_i\alpha_{in})}(q^{-1}) \\ &= q^{-k} \sum_{c_1+\cdots+c_{n-1}=k} q^{-h(\lambda-\sum_i c_i\alpha_{in})}p_n(q)\end{aligned}$$

We make the observation that α_{in} and height $n-i$, so the vector $\lambda - \sum_i c_i\alpha_{in}$ has height $h(\lambda) - \sum_i c_i(n-i)$. Noting that $\sum c_i = k$, the sum becomes:

$$\begin{aligned}&= q^{-k} \sum_{c_1+\cdots+c_{n-1}=k} q^{-h(\lambda-\sum_i c_i\alpha_{in})}p_n(q) \\ &= q^{-k} \sum_{c_1+\cdots+c_{n-1}=k} q^{-[h(\lambda)-\sum_i c_i(n-i)]}p_n(q) \\ &= q^{-h(\lambda)} \sum_{c_1+\cdots+c_{n-1}=k} q^{\sum_i c_i(n-i-1)}p_n(q)\end{aligned}$$

In particular, note that this sum only depends on k, which was defined in section 5.1 to be the last co-ordinate of λ

We return to the rearranged sum at the end of section 5.2:

$$\begin{aligned}\mathfrak{M}_\lambda^0(q^{-1}) &= \sum_{T\subseteq S\setminus S'} (-1)^{|T|} \sum_{D\subseteq S'} (-1)^{|D|}\mathcal{P}\left(\lambda - \sum_{\alpha\in T}\alpha - \sum_{\alpha\in D}\alpha, q^{-1}\right) \\ &= \sum_{T\subseteq S\setminus S'} (-1)^{|T|}q^{-h(\lambda-\sum_{\alpha\in T}\alpha)} \sum_{c_1+\cdots+c_{n-1}=\kappa(\lambda-\sum_{\alpha\in T}\alpha)} q^{\sum_i c_i(n-i-1)}p_n(q) \\ &= q^{-h(\lambda)}p_n(q) \sum_{T\subseteq S\setminus S'} (-1)^{|T|}q^{h(\sum_{\alpha\in T}\alpha)} \sum_{c_1+\cdots+c_{n-1}=\kappa(\lambda-\sum_{\alpha\in T}\alpha)} q^{\sum_i c_i(n-i-1)}\end{aligned}$$

We note further that $S \setminus S'$ is the set of positive roots where the last co-ordinate is -1, so noting that $\kappa(\lambda)$ was defined to be the negative of the final co-ordinate of λ, we have that $\kappa(\lambda - \sum_{\alpha\in T}\alpha) = \kappa(\lambda) - |T|$.

Here are several observations:

1. If the conjecture is true for all n, then we must have that p_n divides p_{n+1}
2. It sufficies to show that for any λ, that the following sum gives a fixed polynomial:
$$\sum_{T\subseteq S\setminus S'} (-1)^{|T|} q^{h(\sum_{\alpha\in T}\alpha)} \sum_{c_1+\cdots+c_{n-1}=\kappa(\lambda)-|T|} q^{\sum_i c_i(n-i-1)}$$
3. The sum above does not in fact depend on all of λ, but only depends on the last co-ordinate of λ. Since we are only summing over the subsets of $S \setminus S'$, which are just the simple roots (not dependent on λ), and the second sum depends only on $\kappa(\lambda) - |T|$, where κ is the negative of the last co-ordinate, so the second sum only depends on the final co-ordinate of λ.

   Since the expansion of the sum only depends on the final co-ordinate, it seems likely that if the original conjecture is true, then the strengthened conjecture should be true too since their decompositions would be the same

8.1 A look into the first inductive step

In the case of $A_3 \subset \mathbb{R}^4$, we consider the possible sets T:

T can be the sets $\emptyset, \{\alpha_{14}\}, \{\alpha_{24}\}, \{\alpha_{34}\}, \{\alpha_{14}, \alpha_{24}\}, \{\alpha_{14}, \alpha_{34}\}, \{\alpha_{24}, \alpha_{34}\}, \{\alpha_{14}, \alpha_{24}, \alpha_{34}\}$.

The final summand $q^{\sum_i c_i(n-i-1)}$ is of the form $q^{2c_1+c_2}$ after substituting $n = 4$.

Note that $\kappa(\lambda) - |T|$ only depends on the size of T, and we can let $-k$ to be the final co-ordinate of λ.

Now we compute the values of $h(\sum_{\alpha\in T}\alpha)$ for our different T.

- If $T = \emptyset$, then the sum $h(\sum_{\alpha\in T}\alpha)$ is simply 0, so the set contributes the following to our sum:
$$\left(\sum_{c_1+c_2+c_3=k} q^{2n_1+n_2}\right)$$
- If $T = \{\alpha_{34}\}$, then this has height 1, and $|T| = 1$, so it contributes the following term to our sum:
$$-q\left(\sum_{c_1+c_2+c_3=k-1} q^{2n_1+n_2}\right)$$
- If $T = \{\alpha_{24}\}$ or $\{\alpha_{14}\}$, we have an identical calculation as above, but the height sum is different, so instead of q in the front we would get q^2 and q^3 respectively.
- If $T = \{\alpha_{14}, \alpha_{24}\}$, the main difference is now that we sum over $c_1 + c_2 + c_3 = k - 2$ now, and the height sum of the set is $3 + 2 = 5$ so we multiply by q^5 at the front.
- We proceed similarly for the rest for the rest of the subsets.

Thus in the case of $n = 4$ our task reduces to showing that:
$$\left(\sum_{c_1+c_2+c_3=k} q^{2n_1+n_2}\right) - \left(\sum_{c_1+c_2+c_3=k-1} q^{2n_1+n_2}\right)(q+q^2+q^3)$$
$$+\left(\sum_{c_1+c_2+c_3=k-2} q^{2n_1+n_2}\right)(q^3+q^4+q^5) - \left(\sum_{c_1+c_2+c_3=k-3} q^{2n_1+n_2}\right)q^6$$
is a fixed polynomial.

This doesn't seem too hard to prove, but the summer project ended before I was able to do so, hopefully if I have some time in the future, I will be able to finish off the induction step.