Vieta Jumping and Polynomials

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Vieta Jumping

Method: Given a solution to a diophantine equation, find a smaller solution using Vieta's formulas.

Problem 1 (IMO 1988)

Let a and b be two positive integers such that $ab+1$ divides a^2+b^2. Show that $\dfrac{a^2+b^2}{ab+1}$ is a perfect square.

Main Idea: Suppose $\frac{a^2+b^2}{ab+1} = k$ for some positive integer k.
If we fix a, then b satisfies the quadratic equation: $x^2 - kax + a^2 - k = 0$
We can find relations between the two roots of this quadratic equation and the coefficients.

IMO 1998 P6

WLOG $a \leq b$.
Suppose k is a non-square, and $a + b$ is minimal such that $\frac{a^2+b^2}{ab+1} = k$.

Let the two solutions to $x^2 - kax + a^2 - k = 0$ are b and b_1. Then:

$$\begin{cases} b + b_1 = ka \\ bb_1 = a^2 - k \end{cases}$$

Thus $b_1 = ka - b = \frac{a^2-k}{b}$, and must be a non-zero integer.

But $b_1 = \frac{a^2-k}{b} < b$ and (a, b_1) also satisfies $\frac{a^2+b_1^2}{ab_1+1} = k$.

From $\frac{a^2+b_1^2}{ab_1+1} = k$ we obtain that b_1 must be positive
This means we have found a smaller solution, contradicting the minimality of $a + b$.

Polynomials: Common Techniques

- Bounding
- Intermediate Value Theorem
- Lagrange Interpolation
- Vieta's Formulas
- Expansion

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Problem 1 (IMO 1988)**.** Let a and b be two positive integers such that $ab+1$ divides a^2+b^2. Show that $\dfrac{a^2+b^2}{ab+1}$ is a perfect square.

Problem 2. Let x and y be positive integers such that xy divides x^2+y^2+1. Show that $\frac{x^2+y^2+1}{xy}=3$.

Problem 3 (IMO 2007)**.** Let a and b be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a=b$.

Problem 4 (IMOSL 2017 N6)**.** Find the smallest positive integer n or show no such n exists, with the following property: there are infinitely many distinct n-tuples of positive rational numbers $(a_1, a_2, \dots, a_n)$ such that both

$$a_1+a_2+\cdots+a_n \quad \text{and} \quad \frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}$$

are integers.

Polynomials

Problem 5 (Russia 2009)**.** Find all value of n for which there are nonzero real numbers a, b, c, d such that after expanding and collecting similar terms, the polynomial $(ax+b)^{100}-(cx+d)^{100}$ has exactly n nonzero coefficients.

Problem 6 (Russia 2002)**.** The polynomials $P(x)$, $Q(x)$, $R(x)$ with real coefficients, one of which is degree 2 and two of degree 3, satisfy the equality $P(x)^2+Q(x)^2=R(x)^2$. Prove that one of the polynomials of degree 3 has three real roots.

Problem 7 (Russia 2003)**.** The side lengths of a triangle are the roots of a cubic polynomial with rational coefficients. Prove that the altitudes of this triangle are roots of a polynomial of sixth degree with rational coefficients.

Problem 8 (Russia 2016)**.** Let n be a positive integer and let $k_0, k_1, \dots, k_{2n}$ be nonzero integers such that $k_0+k_1+\cdots+k_{2n} \neq 0$. Is it always possible to find a permutation $(a_0, a_1, \dots, a_{2n})$ of $(k_0, k_1, \dots, k_{2n})$ so that the equation

$$a_{2n}x^{2n}+a_{2n-1}x^{2n-1}+\cdots+a_0=0$$

has no integer roots?

Problem 9 (Russia 2013)**.** Let $P(x)$ and $Q(x)$ be (monic) polynomials with real coefficients (the first coefficient being equal to 1), and $\deg P(x)=\deg Q(x)=10$. Prove that if the equation $P(x)=Q(x)$ has no real solutions, then $P(x+1)=Q(x-1)$ has a real solution.

Problem 10 (USAMO 2002)**.** Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

Problem 11 (IMO 2016)**.** The equation

$$(x-1)(x-2)\cdots(x-2016)=(x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

Problem 12 (IMO 2006)**.** Let $P(x)$ be a polynomial of degree $n>1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x)=P(P(\dots P(P(x))\dots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t)=t$.