Arithmetic Functions

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1 Arithmetic Functions

Definition 1 (Arithmetic Function)**.** An **Arithmetic function** is a function $\mathbb{N} \to \mathbb{C}$ which *"expresses some arithmetical property of n"* (Hardy & Wright)

Definition 2 (Multiplicative Functions)**.** An arithmetic function f is **multiplicative** if for any coprime $m, n \in \mathbb{N}$, $f(mn) = f(m)f(n)$.

If the above condition is true for ALL (not necessarily coprime) m, n, then we call f **completely multiplicative**.

1.1 Sigma Functions

Definition 3. For some x, define $\sigma_x(n)$ to be the sum of the xth powers of all divisors of n:

$$\sigma_x(n) = \sum_{d|n} d^x$$

In particular, the $\sigma_0(n)$ is the number of positive divisors of n, and is often denoted by $\tau(n)$. σ_1 is often written as simply σ

Question 1. Show that σ_x is multiplicative.

Question 2. If n has prime factorisation $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Find $\sigma_x(n)$ in terms of p_i, α_i.

Question 3. Express $\tau(n)$ in terms of α_i.

1.2 Euler Totient Function

Definition 4 (Euler Totient Function)**.** The Euler Totient Function $\varphi(n)$ gives the number of positive integers less than or equal to n that are coprime to n.

Question 4. Show that φ is multiplicative.

Question 5. If n has prime factorisation $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Find $\varphi(n)$ in terms of p_i, α_i.

Question 6. By considering the fractions $\frac{1}{n}, \frac{2}{n}, \cdots \frac{n}{n}$, show that:

$$\sum_{d|n} \varphi(d) = n$$

2 Dirichlet Convolution

2.1 Motivation

If we have 2 sequences $(a_n), (b_n)$, with generating functions: $A(x) = a_0 + a_1x + a_2x^2 + \cdots$ and $B(x) = b_0 + b_1x + b_2x^2 + \cdots$, if we multiply these 2 generating functions, we obtain:

$A(x)B(x) = C(x) = c_0 + c_1x + c_2x^2 + \cdots$

Where: $c_k = \sum_{i=0}^{n} a_i b_{n-i}$

We say that (c_n) is the convolution of the sequences (a_n) and (b_n)

2.2 Dirichlet Convolutions

If we consider a new kind of generating function (called a Dirichlet Generating Function), defined for an arithmetic function f by:

$$\sum_{n>0} \frac{f(n)}{n^s} = \frac{f(1)}{1^s} + \frac{f(2)}{2^s} + \cdots$$

If we consider the product of two Dirichlet Generating Functions, of arithmetic functions f, g:

$$\left(\frac{f(1)}{1^s} + \frac{f(2)}{2^s} + \cdots\right)\left(\frac{g(1)}{1^s} + \frac{g(2)}{2^s} + \cdots\right) = \left(\frac{h(1)}{1^s} + \frac{h(2)}{2^s} + \cdots\right)$$

Then h is the *Dirichlet Convolution* of f and g, denoted by $h = f * g$, and

$$(f * g)(n) \;=\; \sum_{d\,|\,n} f(d)\, g\!\left(\frac{n}{d}\right) \;=\; \sum_{ab\,=\,n} f(a)\, g(b)$$

Definition 5 (1 function)**.** Define the function $1 : \mathbb{N} \to \mathbb{N}$, where $1(n) = 1$ for each n.

Example 7. $(1 * \varphi)(n) = n$

Definition 6 (Möbius function)**.** Define the function μ, where:

$$\mu(n) = \begin{cases} +1 & \text{if } n \text{ is a square-free positive integer with an even number of prime factors.} \\ -1 & \text{if } n \text{ is a square-free positive integer with an odd number of prime factors.} \\ 0 & \text{if } n \text{ has a squared prime factor.} \end{cases}$$

Question 8. Find $1 * \mu$

Problem 9. Show that if f and g are multiplicative, then $f * g$ is also multiplicative.

3 Möbius Inversion

3.1 Motivation

If we have functions f and g satisfying $g(n) = \sum_{k=0}^{n} f(k)$ for each n. We can find f in terms of g by: $f(n) = g(n) - g(n-1)$.

However, in Number Theory, there are often functions f and g that are related through a relation such as $g(n) = \sum_{d|n} f(d)$ for each n. How can we express f in terms of g in this case?

3.2 Möbius Inversion Formula

Notice that if $g(n) = \sum_{d|n} f(d)$, then $1 * f = g$.

From Question 8 we know that $(1 * \mu)(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$. i.e.

$$\left(\sum_{n>0} \frac{1}{n^s}\right)\left(\sum_{n>0} \frac{\mu(n)}{n^s}\right) = 1$$

So:

$$\begin{aligned} \left(\sum_{n>0} \frac{f(n)}{n^s}\right)\left(\sum_{n>0} \frac{1(n)}{n^s}\right) &= \left(\sum_{n>0} \frac{g(n)}{n^s}\right) \\ \left(\sum_{n>0} \frac{f(n)}{n^s}\right) &= \left(\sum_{n>0} \frac{g(n)}{n^s}\right)\left(\sum_{n>0} \frac{\mu(n)}{n^s}\right) \\ f &= g * \mu \end{aligned}$$

Thus, we have the Möbius Inversion Formula:

Theorem 1 (Möbius Inversion Formula)**.** If f and g are functions such that $g(n) = \sum_{d|n} f(d)$ for each $n \in \mathbb{N}$, then

$$f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right)$$

Question 10. What is $1 * 1$?

Question 11. What is $\tau * \mu$?

Question 12. What is $\text{Id} * 1$? (Id is the identity function: $\text{Id}(n) = n$)

Question 13. What is $\sigma * \mu$?

4 Selected Problems

Problem 14. Prove that for all n: $\sigma(n) + \varphi(n) \geq 2n$

Problem 15 (Slovakia, 2017)**.** Find all natural n for which $\varphi(n)|(n^2 + 3)$.

Problem 16 (Bulgaria, 2019)**.** For a natural number n we denote with $\tau(n)$ the number of all natural divisors of n. Find all numbers n for which, if $1 = d_1 < d_2 < ... < d_k = n$ are all natural divisors of n, then: $\tau(d_1) + \tau(d_2) + ... + \tau(d_k) = \tau(n^3)$ holds.

Problem 17 (CHKMO, 2018)**.** Let k be a positive integer. Prove that there exists a positive integer ℓ with the following property: if m and n are positive integers relatively prime to ℓ such that $m^m \equiv n^n \pmod{\ell}$, then $m \equiv n \pmod{k}$.

Problem 18 (IMOSL 2000 N2)**.** For a positive integer n, let $d(n)$ be the number of all positive divisors of n. Find all positive integers n such that $d(n)^3 = 4n$.

Problem 19 (IMOSL 2016 N2)**.** Let $\tau(n)$ be the number of positive divisors of n. Let $\tau_1(n)$ be the number of positive divisors of n which have remainders 1 when divided by 3. Find all positive integral values of the fraction $\frac{\tau(10n)}{\tau_1(10n)}$.

Problem 20 (IMOSL 2018 N1)**.** Determine all pairs (m, n) of positive integers for which there exists a positive integer s such that sm and sn have an equal number of divisors.